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Correlation Length and Its Critical Exponent for Percolation Processes

Bao Gia Nguyen[1]

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Some critical exponent inequalities are given involving the correlation length of site percolation processes on $\mathbb{Z}^d$. In particular, it is shown that $\nu \geqslant 2/d$, which implies that the critical exponent ν cannot take its mean-field value for the three-dimensional percolation processes.

KEY WORDS: Percolation process; correlation lengths; critical exponent inequalities.

1. INTRODUCTION

We first define the model and introduce the notation we will use in this paper. A site percolation process in $\mathbb{Z}^d$ (here $d \geqslant 2$) is a family of probability measures P_p, $p \in [0, 1]$, together with a collection of random variables $\eta: \mathbb{Z}^d \to \{0, 1\}$ such that under P_p the η's are independent and $P_p(\eta(x) = 1) = p$. A site x is thought of being occupied (nonoccupied) if $\eta(x) = 1$ $[\eta(x) = 0]$. We say that x is connected to y if there is a path of occupied sites connecting x and y; i.e., there is a sequence of sites $x_0 = x$, $x_1, x_2, \ldots, x_n = y$ in $\mathbb{Z}^d$ so that x_i and x_{i+1} are nearest neighbors and $\eta(x_i) = 1$ for every $i = 0, 1, 2, \ldots, n$. We denote this event by $\{x \to y\}$. Let $C_0 = \{x: 0 \to x\}$. We say that C_0 is the cluster containing 0.

It has been shown by Aizenman and Newman[1] that $P_p(0 \to x)$ decays exponentially whenever the site density p is below the critical value

$$p_c = \sup\{p: E_p(|C_0|) < \infty\}$$

[1] Center for Stochastic Processes, University of North Carolina at Chapel Hill, North Carolina.

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This leads to the definition of the correlation length $\xi(p)$ as the minimal value for which

$$P_p(0 \to x) \leqslant \exp[-|x|/\xi(p)], \qquad \text{for all} \quad x \in \mathbb{Z}^d$$

It is easy to see by the FKG inequality that the minimum is attained. Furthermore, one can show that $\xi(p) \uparrow \infty$ as $p \uparrow p_c$. It is of interest to study the rate of decay of the correlation length as $p \uparrow p_c$, which can be represented by the critical exponent ν defined by

$$\nu = -\lim_{p \uparrow p_c} \frac{\log \xi(p)}{\log(p_c - p)}$$

We denote this by $\xi(p) \approx (p_c - p)^{-\nu}$.

As suggested by many physicists, it is believed that the correlation length $\xi(p)$ can be thought of as being the same as the length scales:

$$\xi_t(p) = \left[\sum_x |x|^t \, P_p(0 \to x) \Big/ \sum_x P_p(0 \to x) \right]^{1/t}$$

(see, e.g., Essam[5]). To be more precise, we say that the two length scales $\xi(p)$ and $\xi_t(p)$ are the same if they decay at the same rate; i.e., if we assume that $\xi_t(p) \approx (p_c - p)^{-\nu_t}$, then $\nu = \nu_t$. In support of the above belief we give a proof of the following weaker result and its corollary:

$$0 \leqslant \nu - \nu_t \leqslant (\gamma - \nu)/t \tag{1}$$

where γ is the critical exponent of $E_p(|C_0|)$, i.e.,

$$E_p(|C_0|) \approx (p_c - p)^{-\gamma}$$

Corollary. $\lim_{t \to \infty} \nu_t = \nu$.

In Section 2 we give a proof for the result (1). In the course of doing this we prove some critical exponent inequalities related to scaling theory in Section 3. The scaling theory (see Essam[5]) predicts that

$$P_p(0 \to x) \sim |x|^{-(d-2+\eta)} f(|x|/\xi(p)) \qquad \text{as} \quad p \uparrow p_c \tag{$*$}$$

where $f(r)$ is a function with $f(0) > 0$ and $f(r) \to 0$ exponentially fast as $r \to \infty$, and η is the critical exponent defined by

$$P_{p_c}(0 \to x) \approx |x|^{-(d-2+\eta)} \qquad \text{as} \quad |x| \uparrow \infty$$

Assuming the scaling hypothesis (∗), we can see, by Fubini's theorem, that

$$E_p(|C_0|) = \sum_x P_p(0 \to x) \sim \sum_x |x|^{-(d-2+\eta)} f(|x|/\xi(p))$$
$$= \xi(p)^{2-\eta} \sum_{z = x/\xi(p)} |z|^{-(d-2+\eta)} f(|z|)$$
$$= \text{const} \cdot \xi(p)^{2-\eta} \approx (p_c - p)^{-(2-\eta)\nu}$$

This leads to the critical exponent equality

$$\gamma = (2-\eta)\,\nu$$

As we shall see later, this equality is at least half correct if we assume, for B_n a box of radius n centered at 0, that

$$E_{p_c}(|C_0 \cap B_n|) \equiv \sum_{x \in B_n} P_{p_c}(0 \to x) \approx n^{2-\eta}$$

in replacing the old definition of η as above. (The two definitions for η are expected to be the same. In order for our proof to work, we want to stick with the second definition of η.) In fact, we shall show that it is safe to truncate the sum $E_p(|C_0|) = \sum_x P_p(0 \to x)$ at $n = 2\xi_t(p)$ without losing more than half of the sum, as in the following result:

$$E_p(|C_0|) \geqslant E_p(|C_0 \cap B_n|) \geqslant (1 - 1/2^t)\, E_p(|C_0|) \tag{2}$$

With this bound in hand we immediately see that

$$\gamma \leqslant (2-\eta)\,\nu_t$$

Also in the same section we will show a lower bound for the critical exponent ν: $d\nu \geqslant \Delta_2$, where Δ_2 is defined by

$$E_p(|C_0|^2)/E_p(|C_0|) \approx (p_c - p)^{-\Delta_2} \qquad \text{as} \quad p \uparrow p_c$$

This together with our earlier mean-field bound $\Delta_2 \geqslant 2$ implies that $\nu \geqslant 2/d$. Since the mean-field value of the critical exponent ν is expected to be 1/2 in three dimensions, this critical exponent cannot take its mean-field value.

2. PROOF OF RESULT (1)

Let

$$N(p) = \inf\left\{n\colon \quad \sum_{x:|x|=n} P_p(0 \to x) \leqslant \frac{1}{2}\right\}$$

Aizenman and Newman[1] have shown that

$$N(p) \leqslant 2E_p(|C_0|) \qquad \text{for} \quad p < p_c$$

and also

$$P_p(0 \to x) \leqslant \exp\left(- \frac{\log 2}{N(p)} |x| \right)$$

This shows

$$\xi(p) \leqslant N(p)/2 \tag{3}$$

On the other hand, from the definition of the correlation length

$$\sum_{x:|x|=n} P_p(0 \to x) \leqslant \sum_{x:|x|=n} \exp[-|x|/\xi(p)] \leqslant Kn^{d-1} \exp[-n/\xi(p)]$$

Hence, if $n = d\xi(p) \log \xi(p)$ and if p is close enough to p_c, the rhs will be smaller than 1/2, which gives

$$N(p) \leqslant d\xi(p) \log \xi(p) \tag{4}$$

Thus, by (3) and (4), $N(p)$ and $\xi(p)$ share the same critical exponent. With this in hand we see that

$$\begin{aligned} \xi_t^t E_p(|C_0|) = \sum_x |x|^t P_p(0 \to x) &\geqslant \sum_{x:|x|=N(p)/2}^{N(p)} |x|^t P_p(0 \to x) \\ &\geqslant \sum_{n=N(p)/2}^{N(p)} \left(\frac{N(p)}{2}\right)^t \frac{1}{2} \\ &= \frac{1}{2^{t+2}} N(p)^{t+1} \end{aligned}$$

where in the second inequality we have used the fact that $\sum_{x:|x|=n} P_p(0 \to x) \geqslant 1/2$ if $n \leqslant N(p)$. This leads to the critical inequality

$$t\nu_t + \gamma \geqslant (t+1)\,\nu$$

or

$$(\gamma - \nu)/t \geqslant \nu - \nu_t \tag{5}$$

Furthermore, it is easy to see from Jensen's inequality that ξ_t is increasing in t, hence so is ν_t. Thus, $\lim_{t \to \infty} \nu_t$ exists. Then, letting $t \uparrow \infty$, we get from (5)

$$\nu - \lim_{t \to \infty} \nu_t \leqslant 0$$

To show the other half, we look at

$$\begin{aligned}
\xi_t^t E_p(|C_0|) &= \sum_x |x|^t P_p(0 \to x) \leqslant \sum_x |x|^t \exp[-|x|/\xi(p)] \\
&\leqslant K \sum_{n=0}^{\infty} n^{t+d-1} \exp[-n/\xi(p)] \\
&= K \sum_{l=0}^{\infty} \sum_{l\xi(p) \leqslant n < (l+1)\xi(p)} n^{t+d-1} \exp(-l) \\
&\leqslant K \sum_{l=0}^{\infty} \xi(p)[(l+1)\,\xi(p)]^{t+d-1} \exp(-l) \\
&= K_1[\xi(p)]^{t+d}
\end{aligned}$$

where $K_1 = K\sum_{l=0}^{\infty}(l+1)^{t+d-1}\exp(-l)$. In terms of the critical exponent we have

$$(t+d)\,v \geqslant tv_t + \gamma$$

or

$$v - v_t \geqslant (\gamma - dv)/t \tag{6}$$

Letting $t \uparrow \infty$, from (5) and (6) we get the corollary of result (1) and from this we know that $v \geqslant v_t$, so the result (1) follows. ▮

3. PROOF OF RESULT (2)

Here we show result (2) and derive a lower bound for v. The proof of the result is analogous to Fisher's[6] argument for the Ising model. Observe that

$$\begin{aligned}
E_p(|C_0 \cap B_N|) &\equiv \sum_{x:|x| \leqslant N} P_p(0 \to x) \\
&= \left[1 - \sum_{x:|x|>N} \frac{P_p(0 \to x)}{E_p(|C_0|)}\right] E_p(|C_0|) \\
&\geqslant \left[1 - \sum_{x:|x|>N} \frac{|x|^t P_p(0 \to x)}{N^t E_p(|C_0|)}\right] E_p(|C_0|) \\
&\geqslant \left(1 - \frac{\xi_t^t}{N^t}\right) E_p(|C_0|)
\end{aligned}$$

By choosing $N \geqslant 2\xi_t^t$, we get the second inequality of (2):

$$E_p(|C_0 \cap B_N|) \geqslant (1 - 1/2^t)\, E_p(|C_0|)$$

The other equality of result (2) is trivial. To get a lower bound for ν, we look at

$$\begin{aligned} E_p(|C_0|^2) &= \sum_{x,y} P_p(0 \to x, y) \\ &\leqslant 2 \sum_{x,y:|x| \leqslant |y|} P_p(0 \to x, 0 \to y) \\ &= 2 \sum_x P_p(0 \to x) \sum_{y:|y| \leqslant |x|} P_p(0 \to y \mid 0 \to x) \\ &\leqslant 2K \sum_x |x|^d P_p(0 \to x) \\ &= K' \xi_d^d E_p(|C_0|) \end{aligned}$$

In terms of the critical exponents we have

$$\text{LHS} \approx (p_c - p)^{-\Delta_2 - \gamma}$$
$$\text{RHS} \approx (p_c - p)^{-d\nu_d - \gamma}$$

So

$$\Delta_2 \leqslant d\nu_d$$

But we know that $\Delta_2 \geqslant 2$ (see Durrett and Nguyen[4]), so we have $d\nu \geqslant d\nu_d \geqslant \Delta_2 \geqslant 2$ or $\nu \geqslant 2/d$. ∎

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REFERENCES

1. M. Aizenman and C. Newman, Tree graph inequalities and critical behavior in percolation models, *J. Stat. Phys.* **36**:107 (1984).
2. J. Chayes, L. Chayes, and J. Fröhlich, The low temperature behavior of disordered magnets, *Comm. Math. Phys.* **100**:399 (1985).
3. R. Durrett, Some general results concerning the critical exponents of percolation processes, *Z. Wahrsch. Verw. Geb.* **60**:421 (1985).
4. R. Durrett and B. Nguyen, Thermodynamic inequalities for percolation, *Commun. Math. Phys.* **99**:253 (1985).

5. J. W. Essam, Percolation theory, *Rep. Prog. Phys.* **43**:833–911 (1980).
6. M. E. Fisher, Rigorous inequalities for critical point correlation exponents, *Phys. Rev.* **180**:594–600 (1969).
7. H. Kesten, Analyticity properties and power law estimates of functions in percolation theory, *J. Stat. Phys.* **25**:717–756 (1981).
8. A. Sokal, More inequalities for critical exponents, *J. Stat. Phys.* **25**:25–51 (1981).
9. D. Stauffer, Scaling theory of percolation clusters, *Phys. Rep.* **54**:1–74 (1979).
10. Amit Sur, J. L. Lebowitz, J. Marro, M. H. Kalos, and S. Kirkpatrick, Monte Carlo studies of percolation phenomena for a simple cubic lattice, *J. Stat. Phys.* **19**:345 (1976).